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## Balanced generic circuits without long paths

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## ABSTRACT

We call a graph  $G = (V, E)$  a  $(k, \ell)$ -circuit if  $|E| = k|V| - \ell + 1$  and every  $X \subset V$  with  $2 \leq |X| \leq |V| - 1$  induces at most  $k|X| - \ell$  edges. A  $(2, 3)$ -circuit is also called a *generic circuit*. We say that a graph is *balanced* if the difference between its maximum and minimum degrees is at most 1. Graver et al. asked in Graver et al. (1993) [7] whether a balanced generic circuit always admits a decomposition into two disjoint Hamiltonian paths. We show that this does not hold, moreover there are balanced  $(k, k + 1)$ -circuits for all  $k \geq 2$  which do not contain any Hamiltonian path nor a path longer than  $|V|^\lambda$  for  $\lambda > \frac{\log 8}{\log 9} \simeq 0, 9464$ .

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## 1. Introduction

All graphs considered are undirected and simple (i.e., may not contain loops or multiple edges). Let  $G = (V, E)$  be a graph, and  $k, \ell$  nonnegative integers. Let  $i(X)$  denote the number of edges induced by a subset  $X$  of  $V$ .  $G$  is called  $(k, \ell)$ -sparse if  $i(X) \leq k|X| - \ell$  for every subset  $X \subseteq V$  with  $|X| \geq 2$ . It can easily be checked that the  $(k, \ell)$ -sparse subgraphs of a given graph form the independent sets of a matroid, which is called a  $(k, \ell)$ -count-matroid (see [19]). The circuits of these matroids, that are the minimal no  $(k, \ell)$ -sparse graphs, are the graphs with exactly  $k|V| - \ell + 1$  edges whose every proper subgraph is  $(k, \ell)$ -sparse. Let  $K_n$  denote the complete graph on  $n$  vertices. If  $G$  is a circuit in the  $(k, \ell)$ -count-matroid of  $K_{|V|}$  we call it a  $(k, \ell)$ -circuit. Following [1] we call a  $(2, 3)$ -circuit a *generic circuit*. The term is based on the fact that by Laman's theorem [13],  $(2, 3)$ -circuits are isomorphic to the circuits of the 2 dimensional generic rigidity matroid.

A well-known result of Nash-Williams [14] says that a graph  $G$  is decomposable into  $k$  spanning forests if and only if  $i(X) \leq k(|X| - 1)$  for all nonempty subsets  $X \subseteq V$ . From this theorem it follows that every generic circuit is decomposable into two spanning trees, moreover we obtain the following proposition.

**Proposition 1.1.** *A graph  $G$  is a generic circuit if and only if it can be decomposed into two spanning trees such that no pair of proper subtrees, except single vertices, spans the same vertex set.*

Having this result one may ask whether a tree decomposition with some special properties exists. Graver et al. posed the following problem.

**Open Question 1.2** ([7], Exercise 4.69). *Does every generic circuit with vertices of degrees 3 and 4 only, have a two tree decomposition into two paths?*

Note that the smallest  $(k, k + 1)$ -circuit is  $K_{2k}$  since a graph with  $k|V| - k$  edges cannot be simple if it has less than  $2k$  vertices. Recall the definition of balanced graphs from the abstract. A balanced  $(k, k + 1)$ -circuit has vertices with degrees  $2k - 1$  and  $2k$  only and the number of vertices with degree  $2k - 1$  is exactly  $2k$ .

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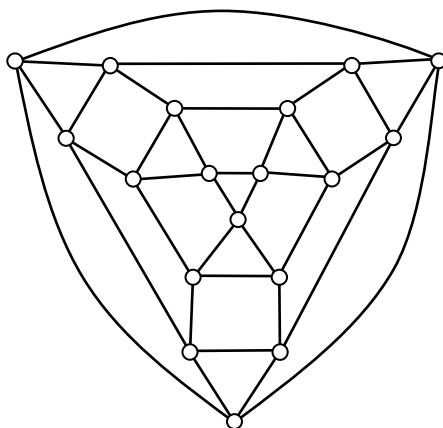


Fig. 1. The graph given by Grünbaum and Malkevitch.

In this note we show balanced  $(k, k + 1)$ -circuits for all  $k \geq 2$  which do not contain a Hamiltonian path. Thus a balanced generic circuit does not always have a decomposition demanded in [Open Question 1.2](#). Moreover, we have a stronger result on the length of the longest paths in balanced  $(k, k + 1)$ -circuits. For a graph  $G$ , let  $h(G)$  and  $h^*(G)$  denote the length of a maximum cycle and the number of vertices in a maximum path of  $G$ , respectively. Following [8] we define the *shortness exponent*  $\sigma(\mathcal{G})$  for a family  $\mathcal{G}$  of graphs, as follows.

$$\sigma(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\log h(G)}{\log |V(G)|}.$$

Concerning paths instead of cycles we also define the parameter  $\sigma^*(\mathcal{G})$  in a similar way but with  $h^*(G)$  in place of  $h(G)$ . Let  $\mathcal{C}_{k,k+1}^{bal.}$  denote the family of balanced  $(k, k + 1)$ -circuits. What we prove is as follows.

**Theorem 1.3.**  $\sigma^*(\mathcal{C}_{k,k+1}^{bal.}) \leq \frac{\log 8}{\log 9}$  for all  $k \geq 2$ .

## 2. Preliminaries

Let  $G = (V, E)$  be a graph. For  $X, Y \subseteq V$ , let  $E(X, Y)$  denote the set of edges with one endpoint in  $X$  and another in  $Y$ . For a subset  $X \subset V$ ,  $E(X, V - X)$  is the *edge-cut* corresponding to  $X$  and  $d_G(X) := |E(X, V - X)|$ . We also say that  $X$  corresponds to the edge-cut. If  $2 \leq |X| \leq |V| - 2$  then the edge-cut is *nontrivial*. We say that a graph  $G$  is *essentially  $k$ -edge-connected*, if all nontrivial edge-cuts of  $G$  contain at least  $k$  edges.

Our construction will be based on the observation that balanced  $(k, k + 1)$ -circuits are just the  $2k$ -regular essentially  $(2k + 2)$ -edge-connected graphs minus a vertex in the following sense.

**Lemma 2.1.** (i) Let  $G = (V, E)$  be a balanced  $(k, k + 1)$ -circuit. If we add a new vertex  $s$  to  $G$  and connect it to the vertices of  $G$  with degree  $2k - 1$  then the obtained graph  $G' = (V', E')$  is  $2k$ -regular and essentially  $(2k + 2)$ -edge-connected.  
(ii) Let  $G' = (V', E')$  be a  $2k$ -regular essentially  $(2k + 2)$ -edge-connected graph. If we omit an arbitrary vertex  $s$  of  $G'$  then the obtained graph  $G = (V, E)$  is a balanced  $(k, k + 1)$ -circuit.

**Proof.** (i) It is clear that  $G'$  is  $2k$ -regular. Suppose that it is not essentially  $(2k + 2)$ -edge-connected. Then there is a subset  $X \subseteq V'$ ,  $2 \leq |X| \leq |V'| - 2$  with  $d_{G'}(X) \leq 2k$ . (As  $G'$  is an Eulerian graph  $d_{G'}(X) = 2k + 1$  cannot hold.) We may assume that  $s \notin X$ . Then  $2i_G(X) = 2i_{G'}(X) = (\sum_{v \in X} d_{G'}(v)) - d_{G'}(X) \geq 2k|X| - 2k$ , a contradiction.

(ii) Let  $X \subseteq V$  be a subset with  $2 \leq |X| \leq |V| - 1$ . Then the same calculation as above admits that  $i_G(X) = i_{G'}(X) \leq k|X| - k - 1$ .  $\square$

We call the graph  $G'$  obtained from a balanced  $(k, k + 1)$ -circuit  $G$  as described in [Lemma 2.1](#) (i) the *underlying regular graph* of  $G$ .

In the case of balanced generic circuits we obtain 4-regular essentially 6-edge-connected graphs. Now we can easily prove that not all balanced generic circuits can be decomposed into two Hamiltonian paths. It is clear that if a balanced generic circuit  $G$  admits such a decomposition, then the four end-vertices of the two Hamiltonian paths must be disjoint and can only be the four vertices with degree 3. Thus in the underlying 4-regular graph  $G'$  they extend to a decomposition into two Hamiltonian cycles. Therefore, it is sufficient to show a 4-regular essentially 6-edge-connected graph which does not have a decomposition into two Hamiltonian cycles.

Grünbaum and Malkevitch [9] gave an example for a 4-regular 4-connected planar graph without a Hamiltonian decomposition (see [Fig. 1](#)). One can easily check that it is essentially 6-edge-connected therefore this is a good example

for our problem as well.<sup>1</sup> (For an elegant proof that this graph is not decomposable, using the fact that it is the medial graph of the Herschel graph, see [3].)

### 3. 3-regular 3-connected graphs without long paths

In our further constructions we will use 3-regular 3-connected graphs as a starting point. In this section we review some of their properties. Let us denote the set of these graphs with  $\mathcal{B}$ .

Examining their shortness parameters Bondy and Simonovits [5] constructed graphs certifying that  $\sigma(\mathcal{B}) \leq \frac{\log 8}{\log 9} \simeq 0.9464$  and Jackson [10] gave the lower bound  $\sigma(\mathcal{B}) \geq \log_2(1 + \sqrt{5}) - 1 \simeq 0.6942$ . The lower bound was recently improved in [2] to  $\lambda \simeq 0.753$ , where  $\lambda$  is the real root of  $4^{1/\lambda} - 3^{1/\lambda} = 2$ . Now considering the case of longest paths we have the same lower bound for  $\sigma^*(\mathcal{B})$  as  $h^*(G) \geq h(G)$  for any graph  $G$ . On the other hand Bondy and Locke [4] showed that  $h^*(G) \leq \frac{3}{2}h(G) - 2$  for all 3-regular 3-connected graphs  $G$ , which implies  $\sigma^*(\mathcal{B}) = \sigma(\mathcal{B})$ . We can also observe that the construction of Bondy and Simonovits actually works for paths as well. The fact that a graph constructed in such a way does not contain a Hamiltonian path either was also used by Singleton [18]. In what follows we will briefly sketch the construction.

First we need some additional notation. For a graph  $G = (V \cup X, E)$  with  $V \cap X = \emptyset$ , we will denote by  $G[X]$  the subgraph of  $G$  induced by  $X$ . The edge set of  $G[X]$  will be denoted by  $E_G(X)$ . Let  $s$  be a new vertex. We say that the (multi)graph  $G' = (V + s, E')$  is obtained from  $G$  by *contracting*  $X$  into  $s$  if  $E'$  is obtained from  $E$  by deleting the edges  $E_G(X)$  and replacing all edges in  $\{uv : u \in X, v \in V\}$  with an edge  $sv$ . Note that this definition admits  $G'$  to have multiple edges but in our case the obtained graphs will be simple. We will also denote with  $G/X$  the graph obtained from  $G$  by contracting  $X$ . To well define the inverse operation we need to specify from which edges of  $G$  are obtained the edges of  $G'$ . Let  $F$  be a graph on the vertex set  $X + t$  with  $t \notin V \cup X + s$ . We say that  $G$  is obtained from  $G'$  by *inflating*  $(F, X)$  into  $s$  if  $G/X = G'$  and  $G/V = F$ .

Considering 3-regular 3-connected graphs we can establish some special properties of these operations. First observe that if  $G$  is 3-regular then it is 3-connected if and only if it is 3-edge-connected.

**Proposition 3.1** ([10]). *Let  $G = (V, E)$  be a 3-regular 3-connected graph and let  $C = E(X, V - X)$  be a nontrivial edge-cut of size 3. Then  $G/X$  and  $G/(V - X)$  are also 3-regular 3-connected (simple) graphs.*

**Proof.** They are simple because  $C$  consists of 3 independent edges by the 3-(vertex)-connectivity of  $G$ . The 3-connectivity follows from the fact that any edge-cut in  $G/X$  or  $G/(V - X)$  is also an edge-cut in  $G$ .  $\square$

**Proposition 3.2.** *Let  $G'$  and  $F$  be 3-regular 3-connected graphs on vertex sets  $V + s$  and  $X + t$  respectively. Let  $G$  be a graph on  $V \cup X$  obtained from  $G'$  by inflating  $(F, X)$  into  $s$ . Then  $G$  is also 3-regular and 3-connected.*

**Proof.** First observe that  $G[X] = F[X]$  and  $G[V] = G'[V]$  are 2-connected because they are both obtained from a 3-connected graph by a deletion of a vertex. Now let  $u, v \in G$  be different vertices. We need to show that  $G - \{u, v\}$  is connected. If  $\{u, v\} \subseteq V$  then  $(G - \{u, v\})/X$  is connected by the 3-connectivity of  $G'$ . Since  $F - t$  is connected, inflating  $(F, X)$  into  $s$  preserves connectivity. The case  $\{u, v\} \subseteq X$  is similar. If  $|\{u, v\} \cap V| = 1$ , say  $u \in V$  and  $v \in X$  then  $G - \{u, v\}$  consists of the two connected subgraphs  $G[V] - u$  and  $G[X] - v$  and at least one edge connecting them.  $\square$

Let  $G_p = (V_p, E_p)$  be the Petersen graph and let  $t$  be an arbitrary vertex of  $G_p$ . Now we can describe the sequence of graphs  $S_0, S_1, \dots, S_i, \dots$  constructed by Bondy and Simonovits. Let  $S_0 = G_p$  and if  $S_i$  is already constructed let  $S_{i+1}$  be obtained from  $S_i$  by inflating  $(G_p, V_p - t)$  into each vertex (see Fig. 2). By Proposition 3.2 all  $S_i$  are 3-connected. Observe that if for a graph  $G = (V \cup X, E)$ ,  $G/X \cong G_p$  and  $P$  is a path in  $G$  with both of its ends in  $X$ , then  $P$  avoids at least one vertex of  $V$ . Otherwise,  $P/X$  is a Hamiltonian cycle in  $G/X$  which is isomorphic to the Petersen graph. Hence  $h^*(S_{i+1}) \leq 8h^*(S_i) + 2$ . Since  $|V(S_i)| = 10 \cdot 9^i$  finally we get  $\lim_{i \rightarrow \infty} \frac{\log h^*(S_i)}{\log |V(S_i)|} = \frac{\log 8}{\log 9}$ .

Now we consider other properties of 3-regular 3-connected graphs. According to Petersen's theorem [16] all these graphs have a perfect matching. Moreover, by Schönberger's result [17] all edges of the graph are included in some perfect matching. We call a perfect matching  $M$  *admissible* if  $|M \cap C| \leq 1$  for all edge-cuts  $C$  of size 3. The following lemma is a consequence of a result of Kaiser and Škrekovski [11] where the statement's complement is proved. For completeness we give here a direct proof.

**Lemma 3.3.** *Let  $G = (V, E)$  be a 3-regular 3-connected graph and let  $e \in E$  be an arbitrary edge of  $G$ . Then  $G$  has an admissible perfect matching which includes  $e$ .*

**Proof.** If the size of every nontrivial edge-cut of  $G$  is greater than 3 then every perfect matching is admissible. Now suppose that there is a nontrivial edge-cut  $C = E(X, V - X)$  of size 3. By Proposition 3.1  $G_1 = G/X$  and  $G_2 = G/(V - X)$  are 3-regular and 3-connected too so by induction they both have an admissible perfect matching including an arbitrary designated edge.

<sup>1</sup> Shortly before finishing this paper we learned that this result was observed independently in [12].

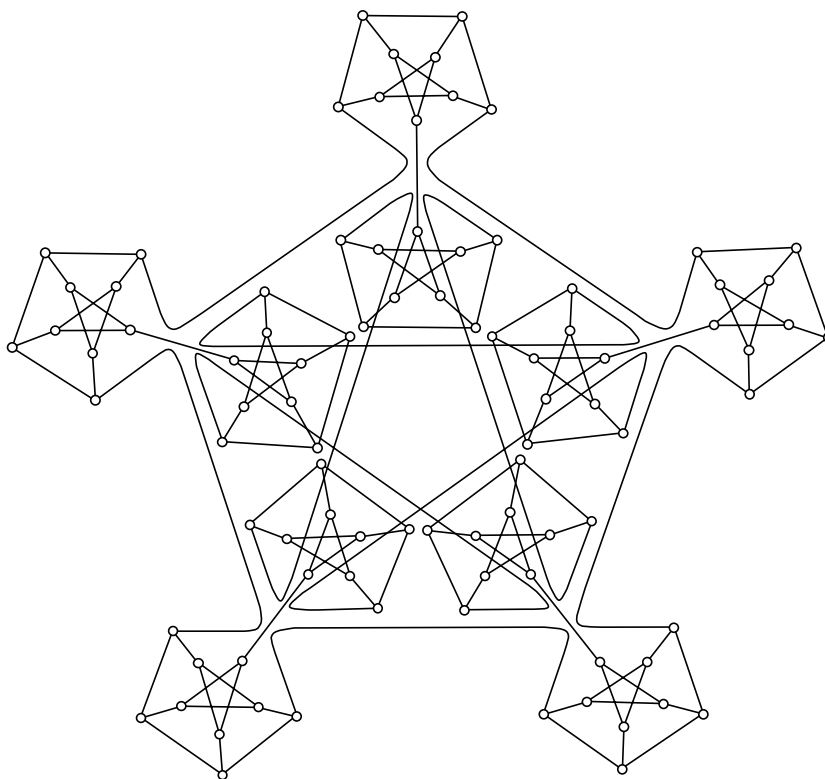


Fig. 2. The graph  $S_1$  constructed by Bondy and Simonovits.

Case 1.  $e \in C$ . Consider an admissible perfect matching  $M_i$  in  $G_i$  including  $e$  for  $i = 1, 2$ . We claim that the union  $M$  of  $M_1$  and  $M_2$  (with one copy of  $e$ ) is an admissible perfect matching in  $G$ . To prove this let  $D$  be an edge-cut with the corresponding vertex-set  $Z$ . If  $Z$  or  $V - Z$  is a subset of  $X$  or  $V - X$  then  $D$  is also an edge-cut in  $G_1$  or  $G_2$  so it does not violate admissibility. Now suppose that  $D$  cuts both  $X$  and  $V - X$ . As we already observed in the proof of Proposition 3.2,  $G[X]$  and  $G[V - X]$  are 2-connected subgraphs. So  $|D| \geq |E(X \cap Z, X - Z)| + |E((V - X) \cap Z, (V - X) - Z)| \geq 2 + 2 = 4$ .

Case 2.  $e \notin C$ . We may assume by symmetry that  $e \in I(V - X)$ . Now consider an admissible perfect matching  $M_1$  in  $G_1$  which includes  $e$ . Let  $f$  be the edge in  $M_1$  incident with the vertex in  $G_1$  corresponding to  $X$ . Now let  $M_2$  be an admissible perfect matching in  $G_2$  which includes  $f$ . Just as in case 1 the perfect matching combining  $M_1$  and  $M_2$  is admissible in  $G$ .  $\square$

#### 4. Our construction

Now we present our construction of balanced  $(k, k + 1)$ -circuits which do not contain long paths. It is clear that  $h^*(G) \leq h^*(G')$  for a  $(k, k + 1)$ -circuit  $G$  and its underlying  $2k$ -regular graph  $G'$ . Therefore, to prove Theorem 1.3 it suffices to show the same proposition for  $2k$ -regular essentially  $(2k + 2)$ -edge-connected graphs by Lemma 2.1.

For the construction we use a special operation similar to inflating that we call *blackberrizing*. This operation ‘substitutes’ all vertices of a  $t$ -regular graph with a special auxiliary graph that we call a *berry*. A berry is a simple graph that consists of two types of vertices that we call *inner vertices* and *outer vertices*, respectively (see Fig. 3). We will use a berry with  $t$  outer vertices for blackberrizing a  $t$ -regular graph  $G = (V, E)$ . We take  $|V|$  disjoint copies of the berry and correspond them to the vertices of  $G$ . If two vertices are adjacent in  $G$  we choose one outer vertex of both corresponding berries and identify them. For the  $t$  neighbors of a vertex we use the  $t$  different outer vertices (in any order). Finally, the inner vertices of a berry correspond to a certain vertex of the basic graph and an outer vertex correspond to an edge of the basic graph (see Fig. 4).

To get  $2k$ -regular graphs by blackberrizing we need that the inner vertices of the berry are of degree  $2k$  and the outer vertices are of degree  $k$ . In our construction we will use the graphs  $S_i$  described in Section 3 as basic graphs.

Since for odd  $k$  there are no graphs with three vertices of degree  $k$  while the other vertices are of degree  $2k$ , we need another operation that we call *parallel blackberrizing*. This operation substitutes the vertices of a  $t$ -regular graph that has a perfect matching. The substitution will be practically as before, but we use the copies of two types of berries. The designated outer vertex of a berry that is substituted to a node  $v$  will be identified with the designated outer vertex of the berry that is substituted to the other endpoint of the matching edge incident to  $v$ . Therefore, when  $k$  is odd we will take a basic graph  $S_i$  (described in Section 3) that has an admissible perfect matching  $M$  by Lemma 3.3; and we will *parallel blackberrize*  $S_i$  along  $M$ .

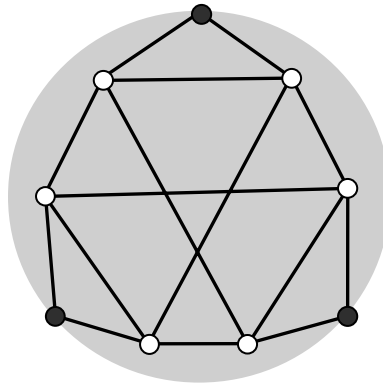


Fig. 3. A copy of a berry which can be used in the case  $k = 2$ . The black vertices are the outer vertices, the white ones are the inner vertices.



Fig. 4. Adjacencies of berries in the blackberry graph.

We will use two berries which have inner vertices of degree  $2k$ , two non-designated outer vertices of degree  $k$  and a designated outer vertex of degrees  $k - 1$  and  $k + 1$ , respectively. Though it is not a requirement of this operation, from now on we will assume that the two berries used in parallel blackberrizing have the same number of vertices.

A graph that arises by (parallel) blackberrizing a regular graph is called a *blackberry graph*. Therefore, in both cases we get a  $2k$ -regular graph  $H$  as blackberry graph.

Examining the properties of this operation we first show that if the basic graph does not contain a long path, then neither does its blackberry graph. The proof is similar to the one in [15].

**Proposition 4.1.** *Let  $S$  be a 3-regular graph and let  $H$  be its blackberry graph. Let  $b$  be the number of vertices in the berries used in the (parallel) blackberrization. Then*

- (i)  $h^*(H) \leq (b - 1) \cdot h^*(S) + 1$ ,
- (ii)  $h^*(H) \leq |V(H)| - (|V(S)| - h^*(S)) (b - 3)$ .

**Proof.** Take an arbitrary path  $P$  in  $H$  and let us consider the sequence of its edges  $e_1, e_2, e_3, \dots, e_m$ . To this sequence we associate a sequence of vertices of  $S$  in the following way: we replace every edge by the vertex of  $S$  which the berry including the edge corresponds to. If the resulting sequence contains identical neighboring elements then we delete some until only one remains consecutively. Let  $v_1, v_2, \dots, v_r$  denote the remaining sequence.

It is easy to check that the consecutive members of this sequence are neighbors in  $S$  and thus it describes a walk in  $S$ . We claim that if  $\{i, j\} \cap \{1, r\} = \emptyset$  with  $i < j$  then  $v_i \neq v_j$ . Suppose for contradiction that  $v_i = v_j$ , let  $B$  be the berry corresponding to this vertex of  $S$  and let  $e_p$  and  $e_q$  be the edges of  $H$  corresponding to  $v_{i-1}$  and  $v_{j+1}$ , respectively. Then in the original path in  $H$ , between  $e_p$  and  $e_q$  we enter and leave  $B$  two times, which contradicts that  $B$  has only 3 outer vertices.

Therefore, if a vertex occurs more than once in the sequence  $v_1, \dots, v_r$  then it is  $v_1$  or  $v_r$ . In this case delete  $v_1$  or  $v_r$  both until every vertex occurs exactly once. Now we obtained a path  $P'$  in  $S$ . For any edge  $e$  in  $H$  the vertex of  $S$  corresponding to the berry including  $e$  is present on  $P'$ . We conclude that  $P$  contains edges from at most  $h^*(S)$  berries. Since a path contains at most  $b - 1$  edges from a berry we have (i). On the other hand, if a path does not use any edge from a berry then it avoids all its inner vertices, which implies (ii).  $\square$

Now we show that if we choose appropriate berries the blackberry graph is essentially  $(2k + 2)$ -edge-connected. To obtain this, we prove more general results for (parallel) blackberrizing.

**Lemma 4.2.** Let  $t \geq 3$  and  $\alpha \geq 3$  be two integers and let  $G$  be a 3-connected  $t$ -regular simple graph and let  $\beta$  be a positive integer with  $\beta \leq \frac{\alpha}{3}$ . Let  $B$  be a berry with  $t$  outer vertices. Let  $\mathcal{O}$  and  $\mathcal{I}$  denote the set of the outer and the inner vertices of  $B$  respectively. Assume that  $B$  satisfies the following properties:

- (i)  $d_B(v) = \alpha - \beta$ ,  $\forall v \in \mathcal{O}$ ,
- (ii)  $d_B(v) = 2\alpha - 2\beta$ ,  $\forall v \in \mathcal{I}$ ,
- (iii) if  $X \subseteq \mathcal{I}$ ,  $|X| \geq 2$  then  $d_B(X) \geq 2\alpha$ .
- (iv) if  $X \subseteq \mathcal{I} \cup \mathcal{O}$ ,  $2 \leq |X| \leq |\mathcal{I} \cup \mathcal{O}| - 2$  then  $d_B(X) \geq \alpha + \beta$ .

Then the graph  $H$  obtained from  $G$  by blackberrizing with  $B$  is  $(2\alpha - 2\beta)$ -regular and essentially  $2\alpha$ -edge-connected.

**Lemma 4.3.** Let  $t \geq 3$  and  $\alpha \geq 4$  be two integers. Let  $G$  be a 3-connected  $t$ -regular simple graph with an admissible perfect matching  $M$  and let  $\beta$  be a positive integer with  $\beta \leq \frac{\alpha-1}{3}$ . Let  $B_1$  and  $B_2$  be two berries with  $t$  outer vertices. Let  $\mathcal{O}_i$  and  $\mathcal{I}_i$  denote the set of outer and inner vertices of  $B_i$  respectively for  $i = 1, 2$ . Assume that  $B_i$  satisfies the following properties for  $i = 1, 2$ :

- (i)' for the designated outer vertex  $u_i \in \mathcal{O}_i$ :  $d_{B_i}(u_i) = \alpha - \beta + (-1)^i$  and  $d_{B_i}(v) = \alpha - \beta$ ,  $\forall v \in \mathcal{O}_i - \{u_i\}$ ,
- (ii)'  $d_{B_i}(v) = 2\alpha - 2\beta$ ,  $\forall v \in \mathcal{I}_i$ ,
- (iii)' if  $X \subseteq \mathcal{I}_i$ ,  $|X| \geq 2$  then  $d_{B_i}(X) \geq 2\alpha$ .
- (iv)' if  $X \subseteq \mathcal{I}_i \cup \mathcal{O}_i$ ,  $2 \leq |X| \leq |\mathcal{I}_i \cup \mathcal{O}_i| - 2$  then  $d_{B_i}(X) \geq \alpha + \beta + 1$ .

Then the graph  $H$  obtained from  $G$  by parallel blackberrizing along  $M$  with  $B_1$  and  $B_2$  is  $(2\alpha - 2\beta)$ -regular and essentially  $2\alpha$ -edge-connected.

To use the above lemmas the following propositions will be useful.

**Proposition 4.4.** Let  $B$  be a berry with 3 outer vertices. Let  $\mathcal{O}$  and  $\mathcal{I}$  denote the set of the outer and the inner vertices of  $B$  respectively. If  $B$  satisfies properties (i)–(iii) from Lemma 4.2, then it also satisfies property (iv).

**Proposition 4.5.** Let  $B_1$  and  $B_2$  be two berries with 3 outer vertices. Let  $\mathcal{O}_i$  and  $\mathcal{I}_i$  denote the set of outer and inner vertices of  $B_i$  respectively for  $i = 1, 2$ . Assume that  $B_i$  satisfies the properties (i)'–(ii)' from Lemma 4.3 and it satisfies the following property for  $i = 1, 2$ :

- (iii)\* if  $X \subseteq \mathcal{I}_i$ ,  $|X| \geq 2$  then  $d_{B_i}(X) \geq 2\alpha + 2$ .

Then it also satisfies property (iv)' for  $i = 1, 2$ .

We prove Propositions 4.4 and 4.5 parallel. If we need some special argument for the proof of Proposition 4.5 we will put it between brackets.

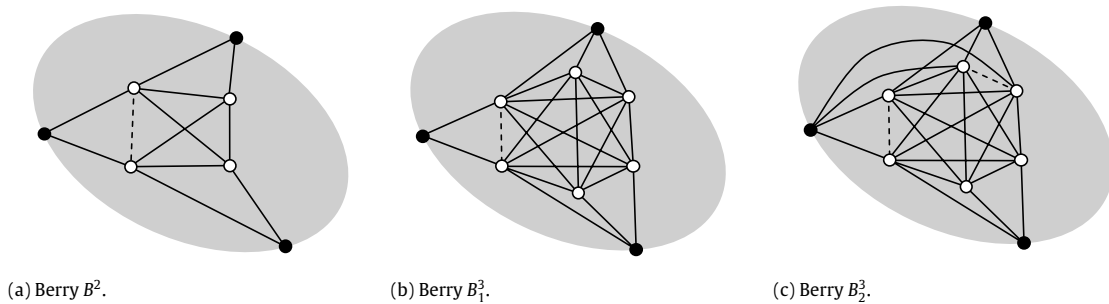
**Proof** (Let  $B := B_i$ ,  $\mathcal{O} := \mathcal{O}_i$ ,  $\mathcal{I} := \mathcal{I}_i$ ). By taking a complement we can assume that  $|X \cap \mathcal{O}| \leq 1$ . If  $|X \cap \mathcal{O}| = 0$ , then (iv)' follows by (iii)\*. Otherwise there are two cases. If  $|X \cap \mathcal{I}| \geq 2$ , then  $d_B(X \cap \mathcal{I}) \geq 2\alpha + 2$  by (iii)\*. Since  $d_B(X) \geq d_B(X \cap \mathcal{I}) - d_B(X \cap \mathcal{O})$ , (iv)' follows by (i)'. If  $|X \cap \mathcal{I}| = 1$ , then  $|X| = 2$ ,  $d_B(X \cap \mathcal{I}) = 2\alpha - 2\beta$  by (ii)' and  $d_B(X \cap \mathcal{O}) \geq \alpha - \beta - 1$  by (i)'. Since the berries are simple graphs  $d_B(X) \geq d_B(X \cap \mathcal{I}) + d_B(X \cap \mathcal{O}) - 2 \geq d_B(X \cap \mathcal{I}) \geq 2\alpha - 2\beta \geq \alpha + \beta + 1$  by (i)'–(ii)',  $\beta \leq \frac{\alpha-1}{3}$  and  $\alpha \geq 3$ .  $\square$

Now we will prove Lemmas 4.2 and 4.3. If we need some special argument for the proof of Lemma 4.3 we will put it between brackets again. In the proof we will also use properties (iv) and (iv)'.

**Proof.** For  $X \subset V(H)$  we call the corresponding edge-cut *proper* if both of  $G[X]$  and  $G[V - X]$  contain a berry. By 3-connectivity of  $G$ , in a proper edge-cut at least 3 berries must be cut. An edge-cut that cuts at least 3 berries includes at least  $3(\alpha - \beta) \geq 2\alpha$  edges by (i)–(iv) and  $\beta \leq \frac{\alpha}{3}$ . [In (i)' it may happen that a cut intersects only  $\alpha - \beta - 1$  edges for a berry  $B$ , namely the edges exiting a designated outer vertex of  $B$ . But then the edge of  $G$  associated with this outer vertex is included in  $M$ . Since  $M$  is admissible there could be just one such a berry-cut if we cut only 3 berries. Hence if we cut 3 berries, then the edge cut includes at least  $3(\alpha - \beta) - 1 \geq 2\alpha$  edges by (i)'–(iv)' and  $\beta \leq \frac{\alpha-1}{3}$ .] If we cut at least 4 berries, then the edge-cut includes at least  $4(\alpha - \beta - 1) \geq 2\alpha$  edges by (i)'–(iv)',  $\beta \leq \frac{\alpha-1}{3}$  and  $\alpha \geq 4$ .] Therefore we only need to consider those cases when the edge-cut is non-proper and it cuts at most two berries.

If a nontrivial edge-cut intersects only one berry  $B'$ , then for one of its corresponding vertex sets, say for  $X$ ,  $X \subseteq \mathcal{I}'$  where  $\mathcal{I}'$  denotes the set of inner vertices of  $B'$ . Therefore since  $|X| \geq 2$ ,  $d(X) \geq 2\alpha$  by (iii)'.





**Fig. 5.** Outer vertices in black, inner vertices in white. The dashed edges are the pinched edges that are not included in the berry.

Now consider the case when a nontrivial edge-cut cuts two berries  $B'$  and  $B''$ . Let  $\mathcal{I}'$ ,  $\mathcal{I}''$ ,  $\mathcal{O}'$  and  $\mathcal{O}''$  be the set of the inner and the outer vertices of  $B'$  and  $B''$  respectively and let  $X$  be the set corresponding to the cut that is included by  $V(B') \cup V(B'')$ . Note that  $X$  contains an outer vertex if and only if it is an outer vertex of both  $B'$  and  $B''$ , because otherwise the edge-cut would cut another berry. Hence if  $|X \cap V(B')| = |X \cap V(B'')| = 1$ , then  $|X \cap \mathcal{I}'| = |X \cap \mathcal{I}''| = 1$  because otherwise  $|X| = 1$ , a contradiction. Therefore in this subcase  $d(X) = 2(2\alpha - 2\beta) \geq 2\alpha$  by (ii)' and  $\beta \leq \frac{\alpha-1}{3}$ . Now by changing the indices we can assume that  $|X \cap V(B')| \geq 2$ . Then by (iv)'  $d_{B'}(X \cap V(B')) \geq \alpha + \beta + 1$ , and by (i)'–(iv)',  $d_{B''}(X \cap V(B'')) \geq \alpha - \beta - 1$ . Hence  $d(X) \geq 2\alpha$ .  $\square$

By *pinching*  $k$  edges of a graph we mean the following operation: take  $k$  independent edges  $u_1 v_1, u_2 v_2, \dots, u_k v_k$  (that is they have no end-vertices in common), delete them, and add a new vertex  $s$  and the  $2k$  edges  $su_1, sv_1, su_2, sv_2, \dots, su_k, sv_k$  to the graph.

Now we define the berries that we use to blackberryize the graphs described in Section 3. For any  $k \geq 2$  we take the complete graph  $K_{2k}$  and we take a partition  $\{V_2, V_3\}$  of its vertices with  $|V_2| = |V_3| = k$ . We add two new vertices  $v_2$  and  $v_3$  and we connect  $v_i$  to all vertices of  $V_i$  for  $i = 1, 2$ . For even  $k$  we pinch  $\frac{k}{2}$  independent edges of  $K_{2k}$  with a new vertex  $v_1$ . We define the resulting graph as the berry  $B^k$  with the outer vertices  $v_1, v_2, v_3$ . (See Fig. 5(a)) For odd  $k$  we pinch  $\frac{k+(-1)^i}{2}$  independent edges of  $K_{2k}$  with a new vertex  $v_1$ . We define the resulting graph as the berry  $B_1^k$  with the designated outer vertex  $v_1$  and non-designated outer vertices  $v_2, v_3$  for  $i = 1, 2$ , respectively. (See Fig. 5(b), (c)) (We note that in the case  $k = 2$  it is easy to find smaller berries, e.g. a triangle which is a berry without any inner vertices works and using this berry the blackberry graph is in fact the line graph of the original one. One can see that if  $k > 2$  the berry holding properties (i)' and (ii)' from Lemma 4.2 [Lemma 4.3] must have at least  $2k$  inner vertices hence there are no simpler ‘good’ berries than the ones described here.)

We show that these berries hold the conditions of Lemma 4.2 [Lemma 4.3].

**Proposition 4.6.** Let  $k \geq 2$  be an even integer. Then the berry  $B := B^k$  holds the conditions of Lemma 4.2 with  $t = 3$ ,  $\alpha = k + 1$  and  $\beta = 1$ .

**Proposition 4.7.** Let  $k \geq 3$  be an odd integer. Then the berries  $B_1 := B_1^k$  and  $B_2 := B_2^k$  hold the conditions of Lemma 4.3 with  $t = 3$ ,  $\alpha = k + 1$  and  $\beta = 1$ .

We give again a parallel proof for the two propositions.

**Proof.** By Proposition 4.4 [Proposition 4.5] it is enough to show that properties (i)', (ii)' and (iii)\* hold. (i)' and (ii)' follows by definition hence we will show only (iii)\*. Let  $X \subseteq \mathcal{I}_{[i]}$  with  $|X| \geq 2$ . We need to show that  $d_{B^k}(X) \geq 2\alpha + 2$ . It is easy to show that  $K_{2k}$  is essentially  $(4k - 4)$ -edge-connected hence  $d_{B^k}(X) \geq d_{K_{2k}}(X) + |X| \geq 4k - 4 + 2 = 4\alpha - 6$ . For  $\alpha = 3$ ,  $4\alpha - 6 = 2\alpha$  and for  $\alpha \geq 4$ ,  $4\alpha - 6 \geq 2\alpha + 2 \geq 2\alpha + 2$ .  $\square$

Now we are ready to prove Theorem 1.3. Let us denote by  $G_i^k$  the graph obtained from the basic graph  $S_i$  by [parallel] blackberryizing it with the berry [berries]  $B^k$  [ $B_1^k$  and  $B_2^k$ ]. [For even  $k$ ,  $|V(B_1^k)| = |V(B_2^k)|$  and  $|\mathcal{I}(B_1^k)| = |\mathcal{I}(B_2^k)|$ ] so we may simply denote these values with  $|V(B^k)|$  and  $|\mathcal{I}(B^k)|$ . By Lemma 4.2 [Lemma 4.3]  $G_i^k$  is  $2k$ -regular and essentially  $(2k+2)$ -edge-connected for all  $i$ . By Proposition 4.1 (i),  $h^*(G_i^k) \leq |V(B^k)| \cdot h^*(S_i)$ . Hence

$$\liminf_{i \rightarrow \infty} \frac{\log h^*(G_i^k)}{\log |V(G_i^k)|} \leq \lim_{i \rightarrow \infty} \frac{\log(|V(B^k)| \cdot h^*(S_i))}{\log(|\mathcal{I}(B^k)| \cdot |V(S_i)|)} = \lim_{i \rightarrow \infty} \frac{\log |V(B^k)| + \log h^*(S_i)}{\log |\mathcal{I}(B^k)| + \log |V(S_i)|} = \lim_{i \rightarrow \infty} \frac{\log h^*(S_i)}{\log |V(S_i)|} = \frac{\log 8}{\log 9}.$$

We note that according to [10] this upper bound cannot be essentially improved by our method that is it remains  $|V|^\lambda$  for a constant  $\lambda > \log_2(1 + \sqrt{5}) - 1$  for any 3-connected 3-regular basic graph and for any berry [berries] for which the conditions of Lemma 4.2 [Lemma 4.3] hold.

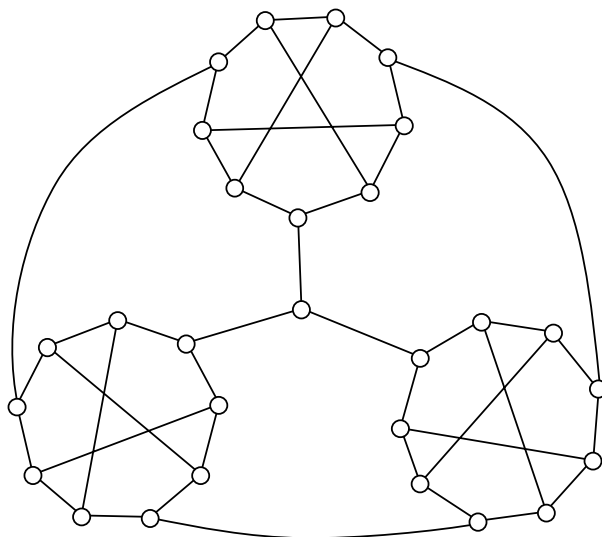
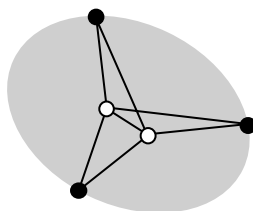


Fig. 6. Singleton-graph.

Fig. 7. A copy of a berry with two adjacent inner vertices which are both adjacent to all 3 outer vertices. This berry can be used in case  $k = 2$ .

## 5. Concluding remarks

### 5.1. Small $(k, k + 1)$ -circuits without long paths

We call the *deficit* of a graph  $G = (V, E)$  the value  $\text{df}(G) = |V(G)| - h^*(G)$ . Our graphs  $G_i^k$  are rather big graphs: even  $S_1$  has 90 vertices before the blackberrization. They proved to be useful for showing [Theorem 1.3](#) but do not provide small examples for  $(k, k + 1)$ -circuits without Hamiltonian paths or with a fixed small deficit. Here we give a method to obtain some small graphs with this property.

First we note that similar to the proof used for the Bondy–Simonovits graphs in [Section 3](#) one can prove that given an arbitrary 3-regular 3-connected graph  $G$  and a positive integer  $t \leq |V(G)|$ , inflating  $(G_p, V_p - s)$  into each of  $t$  vertices of  $G$ ,  $\text{df}(G_p^t) \geq \text{df}(G) + t - 2 + 8 \cdot \max\{t - h^*(G), 0\}$  for the obtained graph  $G_p^t$ . If we choose  $K_4$  as the initial graph and  $t = 3$  we get a 3-regular 3-connected graph  $G$  with only 28 vertices which does not include a Hamiltonian path (see [Fig. 6](#) or [\[18, Figure 5.4\]](#)). We call this graph the Singleton-graph. Using this as the basic graph in the blackberrization we obtain relatively small  $(k, k + 1)$ -circuits without a Hamiltonian path. Counting precisely their deficit is  $|L| - 1$  by [Proposition 4.1 \(ii\)](#).

In the case  $k = 2$  the smallest possible berry including inner vertices is the one with two adjacent inner vertices which are both adjacent to all 3 outer vertices (see [Fig. 7](#)). With this berry we obtain a blackberry graph with 98 vertices and deficit 2 from the Singleton-graph (see [Fig. 8](#)). The resulting generic circuit with 97 vertices is the smallest generic circuit without Hamiltonian paths that the authors know.

Let  $c$  be a positive integer. We give a method for deriving a  $(k, k + 1)$ -circuit with deficit  $\text{df}(G) + c$  from a  $(k, k + 1)$ -circuit  $G$  constructed with berries including inner vertices. First observe that the pinching operation preserves  $2k$ -regularity and also essential  $(2k + 2)$ -edge-connectivity.

**Lemma 5.1.** *Let  $k \geq 2$ . If we pinch  $k$  edges  $u_1v_1, u_2v_2, \dots, u_kv_k$  of a  $2k$ -regular, essentially  $(2k + 2)$ -edge-connected graph  $G = (V, E)$ , then the obtained graph  $G' = (V', E')$  is also  $2k$ -regular and essentially  $(2k + 2)$ -edge-connected.*

**Proof.** Let  $E'(X, V' - X)$  be a nontrivial edge-cut in  $G'$ . We may suppose that the new vertex  $s$  is included in  $V' - X$ . If  $|V' - X| \geq 3$  then  $E(X, V - X)$  is a nontrivial edge-cut in  $G$ , and thus contains at least  $2k + 2$  edges. Suppose that the cut contains  $t$  edges  $u_{i_1}v_{i_1}, \dots, u_{i_t}v_{i_t}$  out of the pinched ones. We may assume that  $u_{i_1}, \dots, u_{i_t} \in X$ . Then  $su_{i_1}, \dots, su_{i_t}$  and the



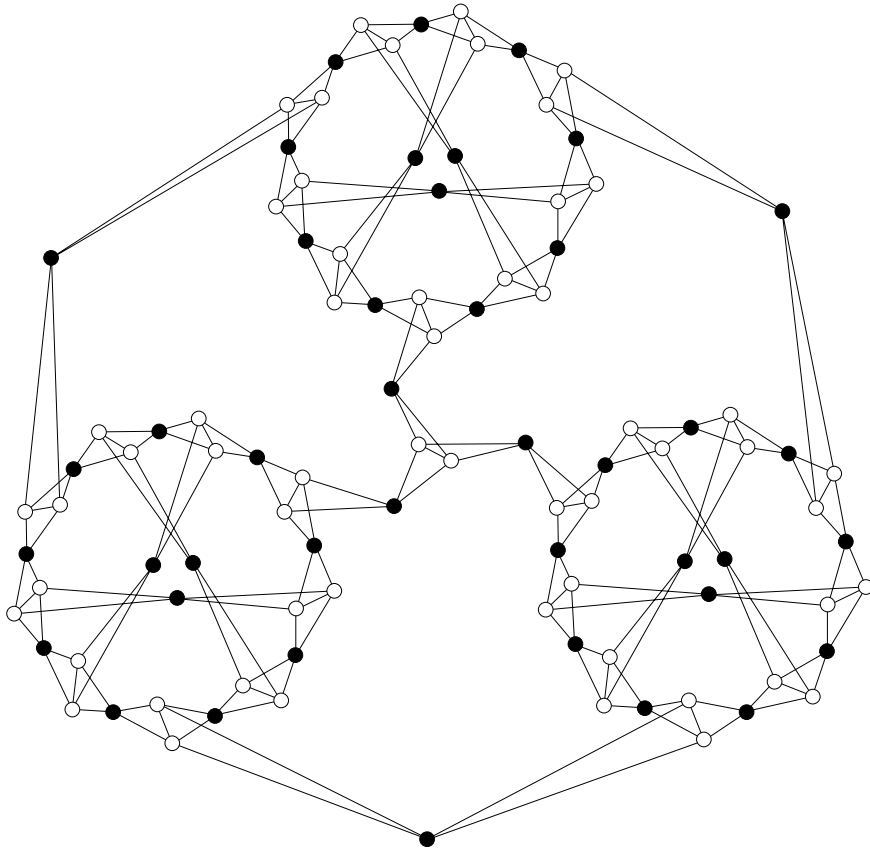


Fig. 8. The blackberry graph of the Singleton-graph.

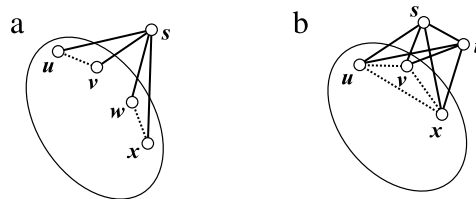


Fig. 9. The two operations used in Theorem 5.2. The dashed edges are deleted from, the solid ones are added to the graph.

$2k + 2 - t$  remaining original edges are  $2k + 2$  edges of the cut in  $G'$ . To complete the proof observe that in a  $2k$ -regular graph for a subset of vertices  $X$  with  $|X| = 2$  the inequality  $d(X) \geq 4k - 2 \geq 2k + 2$  always holds ( $k \geq 2$ ).  $\square$

Now let  $G$  be a blackberry graph. If we successively perform  $c$  arbitrary pinching operations in every berry, the proof of Proposition 4.1 remains valid and we obtain that the lower bound of the deficit increased by  $c$ . The fact that the obtained graph is  $2k$ -regular and essentially  $(2k + 2)$ -edge-connected follows directly from Lemma 5.1 without using Lemmas 4.2 or 4.3.

## 5.2. A constructive characterization

We only showed in Lemma 5.1 that pinching preserves essential  $(2k + 2)$ -edge-connectivity. In fact a much stronger theorem is true in the case  $k = 2$ . It is easy to check that ‘replacing’ a triangle with the berry in Fig. 7 also preserves 4-regularity and essential 6-edge-connectivity. This, together with [6, Lemma 3.2] provides the following constructive characterization:

**Theorem 5.2.** A graph  $G = (V, E)$  is 4-regular and essentially 6-edge-connected if and only if it can be obtained from  $K_5$  by the following operations (see also Fig. 9):

- (a) pinch two independent edges,
- (b) take a triangle  $uvw$  (which means that  $uv, uw, vw$  are all edges), delete its edges and add two new vertices  $s, t$  and edges  $st, su, sv, sw, tu, tv, tw$  to the graph.

This theorem is also an immediate consequence of the constructive characterization of generic circuits given in [1] and of Lemma 2.1.

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